Stable Steady States in Stellar Dynamics

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Abstract

We prove the existence and nonlinear stability of steady states of the Vlasov-Poisson system in the stellar dynamics case. The steady states are obtained as minimizers of an energy-Casimir functional from which fact their dynamical stability is deduced. The analysis applies to some of the well-known polytropic steady states, but it also considerably extends the class of known steady states.

1 Introduction

A galaxy or a globular cluster can be modelled as an ensemble of particles, i. e., stars, which interact only by the gravitational field which they create collectively, collisions among the particles being sufficiently rare to be neglected. In a Newtonian setting the time evolution of such an ensemble is governed by the Vlasov-Poisson system:

$$\partial_t f + v\cdot\nabla_x f - \nabla_x U\cdot\nabla_v f = 0, \tag{1.1}$$

$$\triangle U = 4\pi\rho, \tag{1.2}$$

$$\rho(t,x) = \int f(t,x,v)dv. \tag{1.3}$$

Here $f = f(t,x,v) \geq 0$ denotes the density of the particles in phase space, $t \in \mathbb{R}$ denotes time, $x, v \in \mathbb{R}^3$ denote position and velocity respectively, ρ is the spatial mass density, and U the gravitational potential.

In the present paper we are interested in the existence and stability of steady states of this system. Up to now, steady states for the Vlasov-Poisson system have been contructed in the following way: If U is time independent, the particle energy

$$E = \frac{1}{2}|v|^2 + U(x) \tag{1.4}$$

is conserved along characteristics of (1.1), if U in addition is spherically symmetric the same is true for the modulus of angular momentum

$$L = |x \times v|^2 = |v|^2|x|^2 - (x \cdot v)^2. \tag{1.5}$$

Thus the ansatz

$$f(x,v) = \phi(E,L)$$

satisfies the Vlasov equation and reduces the time independent Vlasov-Poisson system to the semilinear Poisson equation

$$\triangle U = 4\pi h(r,U) \tag{1.6}$$

where

$$h(r,u) = \int \phi\left(\frac{1}{2}|v|^2 + u, L\right) dv,\ r = |x|.$$

In [1] this procedure was carried out for the so-called polytropes

$$\phi(E,L) = (E_0 - E)_+^\mu L^k; \tag{1.7}$$

$(\cdot)_+$ denotes the positive part. The crucial issue there is to show that a solution of (1.6)—once its existence is established—leads to a steady state with finite mass and compact support, cf. also [3]. The approach to this problem used so far is highly dependent on the particular form of ϕ.

In the present paper we construct steady states as minimizers of an appropriately defined energy-Casimir functional. Given a function $Q = Q(f,L) \geq 0,\ f,\ L \geq 0$, we define

$$J(f) := \int\int Q(f,L)\, dv\, dx + \frac{1}{2}\int\int |v|^2 f\, dv\, dx \tag{1.8}$$

and

$$\mathcal{D}(f) := J(f) - \frac{1}{8\pi}\int |\nabla U_f|^2\, dx. \tag{1.9}$$

Here $f = f(x,v)$ is taken from some appropriate set $\mathcal{F}_M$ of functions which in particular have total mass equal to a prescribed constant M and are spherically symmetric, and U_f denotes the potential induced by f with boundary value 0 at spatial infinity. To obtain steady states of the Vlasov-Poisson system as minimizers of the energy-Casimir functional $\mathcal{D}$ has the following advantages: The approach does not rely on a particular form of ansatz like (1.7) so that a broader class of steady states is obtained. The finite mass condition is built into the set of functions $\mathcal{F}_M$ and does not pose an extra problem, and we obtain an explicit bound on the spatial support of the minimizers. Finally and most importantly, the obtained steady states are stable. Throughout we consider only spherically symmetric functions f so that the stability holds with respect to spherically symmetric perturbations.

We now describe in more detail how the paper proceeds. In the next section we state the assumptions for the function Q which determines our energy-Casimir functional, and prove some preliminary results, in particular a lower bound of $\mathcal{D}$ on $\mathcal{F}_M$. The main difficulties in finding a minimizer of $\mathcal{D}$ arise from the fact that $\mathcal{D}$ is neither positive definite nor convex, and from the lack of compactness: Along a minimizing sequence some mass could escape to infinity. This is impossible due to two crucial observations which are established in the third section. The first one is based on a scaling argument and asserts that $\mathcal{D}_M = \inf_{\mathcal{F}_M} \mathcal{D}(f) < 0$ and

$$\mathcal{D}_{M_1} \geq \left(\frac{M_1}{M_2}\right)^{1+\alpha} \mathcal{D}_{M_2}$$

for all positive $M_1 \leq M_2$ and some $\alpha > 0$. The second one is based on a splitting argument. We split the physical space into $B_R = \{|x| \leq R\}$ and its complement. From the above scaling identity we obtain the estimate

$$\mathcal{D}(f) - \mathcal{D}_M \geq \left(-\frac{C_\alpha \mathcal{D}_M}{M^2} - \frac{1}{R}\right) m_f(R)\,(M - m_f(R)).$$

Here $C_\alpha > 0$ is a constant and $m_f(R) = \int_{B_R}\int f$. This implies that along any minimizing sequence the mass has to concentrate in a fixed ball. In the fourth section we use this to show that a minimizer of $\mathcal{D}$ over $\mathcal{F}_M$ exists, and

we prove that every such minimizer is a steady state of the Vlasov-Poisson system. In particular we show that the gravitational field ∇U_f converges strongly in $L^2(\mathbb{R}^3)$ along any minimizing sequence. Dynamical stability of such a minimizer f_0 then follows easily from the fact that $\mathcal{D}$ is conserved along spherically symmetric solutions of the Vlasov-Poisson system. To measure the distance of a perturbation from f_0 we use the quantity

$$\int\!\!\int \Big[Q(f,L)-Q(f_0,L)+(E-E_0)(f-f_0)\Big]\,dv\,dx+\frac{1}{8\pi}\|\nabla U_f-\nabla U_{f_0}\|_2^2;$$

the first term turns out to be nonnegative. Similar constructions have been used in the previous study of stability in collisionless plasmas by the one of the authors [6, 7]. A delicate point arises from the question whether the minimizer is unique in the set $\mathcal{F}_M$. This can be shown in the case of the polytropes. For the general case we had to leave this question open, which results in the fact that we then obtain the stability only with respect to the whole set of minimizers.

We conclude this introduction with some references to the literature. The existence of global classical solutions has been shown in [12] as well as in [10, 11, 15]. The existence of steady states for the case of the polytropic ansatz was investigated in [1] and [3]. There have been many contributions to the stability problem in the astrophysics literature; we refer to the monograph [5]. As far as mathematically rigorous results are concerned, we mention [17], where the stability of the polytropes is investigated using a variational approach for a reduced energy-Casimir functional defined on the space of mass functions $m(r)=4\pi\int_0^r s^2\rho(s)\,ds$, and an investigation of linearized stability in [2]. For the plasma physics case, where the sign in the Poisson equation (1.2) is reversed, the stability problem is much easier and better understood. We refer to [4, 8, 9, 13].

2 Preliminaries; a Lower Bound for $\mathcal{D}$

We first state the assumptions on Q which are needed in the following:

Assumptions on Q: For $Q\in C^{1,0}([0,\infty[\times[0,\infty[)\cap C^{2,0}(]0,\infty[\times[0,\infty[)$, $Q\geq 0$, and constants $C_1,\dots,C_4>0$, $F_0>0$, and $0<\mu_1,\mu_2,\mu_3<3/2$ consider the following assumptions:

(Q1) $Q(f,L)\geq C_1 f^{1+1/\mu_1},\ f\geq F_0,\ L\geq 0,$

(Q2) $Q(f,L) \leq C_2 f^{1+1/\mu_2},\ 0 \leq f \leq F_0,\ L \geq 0,$

(Q3) $Q(\lambda f,L) \geq \lambda^{1+1/\mu_3} Q(f,L),\ f \geq 0,\ 0 \leq \lambda \leq 1,\ L \geq 0,$ and $Q(f,\cdot)$ is decreasing for all $f \geq 0$ if $\mu_3 < 1/2$,

(Q4) $\partial_f^2 Q(f,L) > 0,\ f > 0,\ L \geq 0,$ and $\partial_f Q(0,L) = 0,\ L \geq 0,$

(Q5) $C_3 \partial_f^2 Q(f,L) \leq \partial_f^2 Q(\lambda f,L) \leq C_4 \partial_f^2 Q(f,L)$ for $f > 0,\ L \geq 0$ and λ in some neighborhood of 1.

The above assumptions imply that for fixed $L \geq 0$ the function $\partial_f Q(\cdot,L)$ is strictly increasing with range $[0,\infty[$, and we denote its inverse by $q(\cdot,L)$, i. e.,

$$\partial_f Q(q(e,L),L) = e,\ e \geq 0,\ L \geq 0; \tag{2.1}$$

we extend $q(\cdot,L)$ by $q(e,L)=0,\ e<0$.

Remark: The steady states obtained later will be of the form

$$f_0(x,v) = q(E_0 - E, L)$$

with some $E_0 < 0$ and E and L as defined in (1.4) and (1.5) respectively. If we take $Q(f,L) = f^{1+1/\mu},\ f \geq 0$, this leads to the polytropic ansatz (1.7) with $k=0$, and such a Q satisfies the assumptions above if $0<\mu<3/2$. If we take

$$Q(f,L) = f^{1+1/\mu_1}\psi_1(L) + f^{1+1/\mu_2}\psi_2(L) \tag{2.2}$$

with $0 < \mu_1, \mu_2 < 3/2$ and continuous functions ψ_1, ψ_2 with $0 < c \leq \psi_1(L), \psi_2(L) \leq C < \infty,\ L \geq 0$, both decreasing if $\mu_1 < 1/2$ or $\mu_2 < 1/2$, then again the above assumptions hold, but q is clearly not of polytropic form.

We will minimize the energy-Casimir functional $\mathcal{D}$ over the set

$$\mathcal{F}_M := \Big\{ f \in L^1(\mathbb{R}^6) \mid f \geq 0,\ \iint f\,dv\,dx = M,\ J(f) < \infty, \text{ and } f \text{ is spherically symmetric} \Big\}, \tag{2.3}$$

where $M>0$ is prescribed. Here spherical symmetry means that

$$f(Ax,Av) = f(x,v),\ x,v \in \mathbb{R}^3,\ \mathrm{A} \in \mathrm{SO}(3).$$

The aim of the present section is to establish a lower bound for $\mathcal{D}$ of a form that will imply the boundedness of J along any minimizing sequence. To this end we first establish two technical lemmas:

Lemma 1 *Let $n=3/2+\mu$ and $\mu>0$. Then there exists a constant $C>0$ such that for all measurable $f\geq 0$,*

$$\int \rho_f^{1+1/n} dx \leq C\left(\iint f^{1+1/\mu} dv\,dx + \iint |v|^2 f\,dv\,dx\right)$$

where

$$\rho_f(x) = \int f(x,v)\,dv.$$

Proof. For any $R>0$,

$$\begin{aligned}\rho_f(x) &= \int_{|v|\leq R} f(x,v)\,dv + \int_{|v|\geq R} f(x,v)\,dv\\ &\leq C R^{3/(1+\mu)} \left(\int f^{1+1/\mu} dv\right)^{\mu/(\mu+1)} + \frac{1}{R^2}\int |v|^2 f\,dv\end{aligned}$$

by Hölder's inequality. Optimizing the right hand side with respect to R, taking both sides of the inequality to the power $1+1/n$ and integrating with respect to x yields

$$\begin{aligned}\int \rho_f^{1+1/n} dx &\leq C\int \left[\left(\int f^{1+1/\mu} dv\right)^{2\mu/(5+2\mu)} \left(\int |v|^2 f\,dv\right)^{3/(5+2\mu)}\right]^{1+1/n} dx\\ &\leq C\int \left(\int f^{1+1/\mu} dv + \int |v|^2 f\,dv\right)^{(1+1/n)(3+2\mu)/(5+2\mu)} dx,\end{aligned}$$

and since $1+1/n=(5+2\mu)/(3+2\mu)$ this is the assertion. *QED*

Lemma 2 *Let $\rho\in L^{1+1/n}(\mathbb{R}^3)$ be nonnegative and spherically symmetric with $\int\rho=M$ and $1\leq n<3$. Define*

$$U_\rho := -\frac{1}{|\cdot|}*\rho.$$

(a) $U_\rho\in L^{12}(\mathbb{R}^3)\cap \mathrm{W}^{2,1+1/n}_{\mathrm{loc}}(\mathbb{R}^3)$ with

$$\nabla U_\rho(x) = \frac{x}{r}U_\rho'(r) = \frac{x}{r}\frac{m_\rho(r)}{r^2},\ r=|x|>0,$$

where

$$m_\rho(r) = 4\pi\int_0^r \rho(s)\,s^2 ds = \int_{|x|\leq r}\rho(x)\,dx.$$

(b) For every $R>0$,

$$\int |\nabla U_\rho|^2 dx \le \frac{3n}{3-n}\left(\frac{4\pi}{3}\right)^{1+1/n} M^{1-1/n} R^{(3-n)/n} \int_{|x|\le R} \rho^{1+1/n} dx + \frac{4\pi M^2}{R}.$$

Proof. As to (a) we note that $1+1/n>4/3$ so that $U_\rho \in L^{12}(\mathbb{R}^3)$ by the generalized Young's inequality. The remaining assertions in (a) follow from spherical symmetry. As to (b),

$$\begin{aligned}\int |\nabla U_\rho|^2 dx &\le 4\pi \int_0^R r^{-2} m_\rho^2(r)dr + 4\pi \int_R^\infty r^{-2} m_\rho^2(r)dr \\ &\le 4\pi M^{1-1/n} \int_0^R r^{-2} m_\rho^{1+1/n}(r)dr + \frac{4\pi M^2}{R},\end{aligned}$$

since $m_\rho(r) \le M,\ r\ge 0$. By Hölder's inequality,

$$m_\rho(r) \le \left(\frac{4\pi}{3} r^3\right)^{1/(1+n)} \left(\int_{|x|\le R} \rho^{1+1/n} dx\right)^{n/(1+n)},\ r\le R,$$

and the assertion follows. *QED*

Lemma 3 *Let Q satisfy assumption (Q1). Then there exists a constant C_M depending on M such that*

$$\mathcal{D}(f) \ge \frac{1}{2} J(f) - C_M,\ f \in \mathcal{F}_M,$$

in particular,

$$\mathcal{D}_M := \inf\{\mathcal{D}(f) \mid f\in \mathcal{F}_M\} > -\infty.$$

Proof. Let $f\in\mathcal{F}_M$. By assumption (Q1),

$$\begin{aligned}\iint f^{1+1/\mu_1} dv\,dx &= \iint_{\{f\le F_0\}} \ldots + \iint_{\{f\ge F_0\}} \ldots \le F_0^{1/\mu_1} \iint f + C_1^{-1} \iint Q(f,L) \\ &\le C(M+J(f)).\end{aligned}$$

Thus by Lemma 1,

$$\int \rho^{1+1/n} dx \le C(M+J(f))$$

where $1 \le n = 1 + 1/\mu_1 < 3$. In particular, Lemma 2 applies, and

$$\mathcal{D}(f) \ge J(f)\left(1 - CM^{1-1/n}R^{(3-n)/n}\right) - C\left(M^{2-1/n}R^{(3-n)/n} + \frac{M^2}{R}\right)$$

where $C > 0$ is some constant which does not depend on R. In dependence of M we now choose $R > 0$ such that the term in the first parenthesis equals $1/2$, and the proof is complete. *QED*

3 Scaling and Splitting

The behaviour of $\mathcal{D}$ and M under scaling transformations can be used to relate the $\mathcal{D}_M$'s for different values of M:

Lemma 4 *Let Q satisfy the assumptions (Q2) and (Q3). Then $-\infty < \mathcal{D}_M < 0$ for each $M > 0$, and there exists $\alpha > 0$ such that for all $0 < M_1 \le M_2$,*

$$\mathcal{D}_{M_1} \ge \left(\frac{M_1}{M_2}\right)^{1+\alpha} \mathcal{D}_{M_2}.$$

Proof. Given any function f, we define a rescaled function $\bar{f}(x,v) = af(bx,cv)$, where a, b, $c > 0$. Then

$$\iint \bar{f}\,dv\,dx = ab^{-3}c^{-3}\iint f\,dv\,dx \tag{3.1}$$

and

$$\begin{aligned}\mathcal{D}(\bar{f}) = b^{-3}c^{-3}&\iint Q(af, b^{-2}c^{-2}L)\,dv\,dx \\ &+ ab^{-3}c^{-5}\frac{1}{2}\iint |v|^2 f\,dv\,dx - a^2b^{-5}c^{-6}\frac{1}{8\pi}\int |\nabla U_f|^2\,dx.\end{aligned} \tag{3.2}$$

Proof of $\mathcal{D}_M < 0$: Fix some $f \in \mathcal{F}_1$ with compact support and $f \le F_0$. Let

$$a = Mb^3c^3$$

so that

$$\iint \bar{f}\,dv\,dx = M.$$

The last term in $\mathcal{D}(\bar{f})$ is negative and of the order b, and we want to make this term dominate the others as $b \to 0$. Choose $c = b^{-\gamma/2}$ and assume that $a \leq 1$ so that $af \leq F_0$. By (Q2),

$$\mathcal{D}(\bar{f}) \leq CM^{1+1/\mu_2} b^{\frac{3}{\mu_2}(1-\gamma/2)} + CMb^{\gamma} - M^2 \overline{C} b$$

where C, $\overline{C} > 0$ depend on f. Since we want the last term to dominate as $b \to 0$, we need $\gamma > 1$ and $3(1-\gamma/2)/\mu_2 > 1$, and, in order that $a \leq 1$ as $b \to 0$, also $\gamma < 2$. Such a choice of γ is possible since $\mu_2 < 3/2$, and thus $\mathcal{D}_M < 0$ for b sufficiently small.

Proof of the scaling inequality if $0 < \mu_3 < 1/2$: Assume that $f \in \mathcal{F}_{M_2}$ and $\bar{f} \in \mathcal{F}_{M_1}$ so that by (3.1),

$$ab^{-3}c^{-3} = \frac{M_1}{M_2} =: m \leq 1. \tag{3.3}$$

By (3.2) and (Q3),

$$\mathcal{D}(\bar{f}) \geq m a^{1/\mu_3} \int\!\!\int Q(f,L)\, dv\, dx + mc^{-2} \frac{1}{2} \int\!\!\int |v| f\, dv\, dx - m^2 b \frac{1}{8\pi} \int |\nabla U_f|^2 dx$$

provided $a \leq 1$ and $b^{-2}c^{-2} \leq 1$. Now we require that

$$m a^{1/\mu_3} = mc^{-2} = m^2 b.$$

Together with (3.3) this determines a, b, c in terms of m. In particular,

$$a = m^{4\mu_3/(3-2\mu_3)} \leq 1, \; bc = m^{(1-2\mu_3)/(2\mu_3-3)} \geq 1$$

as required—recall that $0 < \mu_3 < 1/2$ in the present case—and

$$m a^{1/\mu_3} = m^{1+\alpha}, \; \alpha = 4/(3-2\mu_3)$$

whence

$$\mathcal{D}(\bar{f}) \geq m^{1+\alpha} \mathcal{D}(f).$$

Since for any given choice of a, b, c the mapping $f \mapsto \bar{f}$ is one-to-one and onto between $\mathcal{F}_{M_2}$ and $\mathcal{F}_{M_1}$ the scaling inequality follows.

Proof of the scaling inequality if $\mu_3 \geq 1/2$: In this case we choose $a = b = c^{-1}$. If $f \in \mathcal{F}_{M_2}$ and $\bar{f} \in \mathcal{F}_{M_1}$ then again (3.3) holds. Thus $a = m \leq 1$, and since $1 + 1/\mu_3 \leq 3$,

$$\begin{aligned}\mathcal{D}(\bar{f}) &\geq a^{1+1/\mu_3} \int\!\!\int Q(f,L)\, dv\, dx + a^3 \left(\frac{1}{2} \int\!\!\int |v|^2 f\, dv\, dx - \frac{1}{8\pi} \int |\nabla U_f|^2 dx \right) \\ &\geq m^3 \mathcal{D}(f),\end{aligned}$$

which proves the scaling assertion in this case. *QED*

We now prove a splitting estimate which is crucial to find a minimizer of $\mathcal{D}$. We define the ball $B_R = \{x \in \mathbb{R}^3 \mid |\mathrm{x}| \leq \mathrm{R}\}$.

Lemma 5 *Let Q satisfy the assumptions (Q2) and (Q3), let $f \in \mathcal{F}_M$, and*

$$m_f(R) := \int_{B_R} \int f \, dv \, dx, \ R > 0.$$

Then

$$\mathcal{D}(f) - \mathcal{D}_M \geq \left(-\frac{C_\alpha \mathcal{D}_M}{M^2} - \frac{1}{R} \right) m_f(R) (M - m_f(R)), \ R > 0,$$

where the constant $C_\alpha > 0$ depends on α from Lemma 4.

Proof. Let $1_{B_R \times \mathbb{R}^3}$ be the characteristic function of $B_R \times \mathbb{R}^3$,

$$f_1 = 1_{B_R \times \mathbb{R}^3} f, \ f_2 = f - f_1$$

and let ρ_i and U_i denote the induced spatial densities and potentials respectively, $i = 1,2$. We abbreviate $\lambda = M - m_f(R)$. Then

$$\begin{aligned} \mathcal{D}(f) &= J(f_1) + J(f_2) - \frac{1}{8\pi} \int |\nabla U_1|^2 - \frac{1}{8\pi} \int |\nabla U_2|^2 - \frac{1}{4\pi} \int \nabla U_1 \cdot \nabla U_2 \\ &\geq \mathcal{D}_{M-\lambda} + \mathcal{D}_\lambda - \frac{1}{4\pi} \int \nabla U_1 \cdot \nabla U_2. \end{aligned}$$

since $f_1 \in \mathcal{F}_{M-\lambda}$ and $f_2 \in \mathcal{F}_\lambda$. By Lemma 2 (a), $\nabla U_2 = 0$ on B_R, and

$$\int \nabla U_1 \cdot \nabla U_2 \leq \lambda (M - \lambda) 4\pi \int_R^\infty \frac{1}{r^2} dr = \frac{4\pi}{R} \lambda (M - \lambda).$$

Using Lemma 4 we find that

$$\mathcal{D}(f) \geq \left[(1 - \lambda/M)^{1+\alpha} + (\lambda/M)^{1+\alpha} \right] \mathcal{D}_M - \frac{1}{R} \lambda (M - \lambda).$$

Since $\alpha > 0$, there is $C_\alpha > 0$, such that

$$(1 - x)^{1+\alpha} + x^{1+\alpha} - 1 \leq -C_\alpha (1 - x) x, \ 0 \leq x \leq 1.$$

Choosing $x = \lambda/M$ and noticing that $\mathcal{D}_M < 0$, we have

$$\begin{aligned}
\mathcal{D}(f) - \mathcal{D}_M &\geq \left[(1-\lambda/M)^{1+\alpha} + (\lambda/M)^{1+\alpha} - 1\right] \mathcal{D}_M - \frac{1}{R}\lambda(M-\lambda) \\
&\geq -C_\alpha \mathcal{D}_M \left(1 - \frac{\lambda}{M}\right) \frac{\lambda}{M} - \frac{1}{R}\lambda(M-\lambda) \\
&= \left(-\frac{C_\alpha \mathcal{D}_M}{M^2} - \frac{1}{R}\right)(M-\lambda)\lambda,
\end{aligned}$$

and the proof is complete. *QED*

4 Minimizers of $\mathcal{D}$

Before we show the existence of a minimizer of $\mathcal{D}$ over the set $\mathcal{F}_M$ we use Lemma 5 to show that along a minimizing sequence the mass has to concentrate in a certain ball:

Lemma 6 *Let Q satisfy the assumptions (Q2) and (Q3), and define*

$$R_0 := -\frac{M^2}{C_\alpha \mathcal{D}_M}.$$

If $(f_n) \subset \mathcal{F}_M$ is a minimizing sequence of $\mathcal{D}$, then for any $R > R_0$,

$$\lim_{n\to\infty} \int_{|x|\geq R} \int f_n dv\, dx = 0.$$

Proof. If not, there exist some $R > R_0$, $\lambda > 0$, and a subsequence, called (f_n) again, such that

$$\lim_{n\to\infty} \int_{|x|\geq R} \int f_n dv\, dx = \lambda.$$

For every $n \in \mathbb{N}$ we can now choose $R_n > R$ such that

$$\lambda_n := \int_{|x|\geq R_n} \int f_n dv\, dx = \frac{1}{2} \int_{|x|\geq R} \int f_n dv\, dx.$$

Then

$$\lim_{n\to\infty} \int_{|x|\geq R_n} \int f_n dv\, dx = \lim_{n\to\infty} \lambda_n = \lambda/2 > 0.$$

Applying Lemma 5 to B_{R_n} we get

$$\begin{aligned}\mathcal{D}(f_n)-\mathcal{D}_M &\geq \Big(-\frac{C_\alpha \mathcal{D}_M}{M^2}-\frac{1}{R_n}\Big)(M-\lambda_n)\lambda_n > \Big(-\frac{C_\alpha \mathcal{D}_M}{M^2}-\frac{1}{R}\Big)(M-\lambda_n)\lambda_n \\ &\to \Big(-\frac{C_\alpha \mathcal{D}_M}{M^2}-\frac{1}{R}\Big)(M-\lambda/2)\lambda/2>0\end{aligned}$$

as $n\to\infty$, since by choice of R_0 the expression in the parenthesis is positive for $R>R_0$, and $0<\lambda/2<M$. This contradicts the fact that (f_n) is a minimizing sequence. *QED*

Theorem 1 *Let Q satisfy the assumptions (Q1)–(Q4), and let $(f_n)\subset\mathcal{F}_M$ be a minimizing sequence of $\mathcal{D}$. Then there is a minimizer f_0 and a subsequence (f_{n_k}) such that $\mathcal{D}(f_0)=\mathcal{D}_M$,* $\mathrm{supp} f_0\subset B_{R_0}\times \mathbb{R}^3$ *with R_0 as in Lemma 6, and $f_{n_k}\rightharpoonup f_0$ weakly in $L^{1+1/\mu_1}(\mathbb{R}^6)$. For the induced potentials we have $U_{n_k}\to U_0$ strongly in $L^2(\mathbb{R}^3)$.*

Proof. By Lemma 3, $(J(f_n))$ is bounded. Let $p_1=1+1/\mu_1$. By assumption (Q1),

$$\iint f_n^{p_1}dv\,dx\leq C\iint f_n dv\,dx+C\iint Q(f_n,L)\,dv\,dx$$

so that (f_n) is bounded in $L^{p_1}(\mathbb{R}^6)$. Thus there exists a weakly convergent subsequence, denoted by (f_n) again, i. e.,

$$f_n\rightharpoonup f_0 \text{ weakly in } L^{p_1}(\mathbb{R}^6).$$

Clearly, $f_0\geq 0$ a. e., and f_0 is spherically symmetric. Since

$$\begin{aligned}M &= \lim_{n\to\infty}\int_{|x|\leq R_1}\int_{|v|\leq R_2} f_n dv\,dx+\lim_{n\to\infty}\int_{|x|\leq R_1}\int_{|v|\geq R_2} f_n dv\,dx \\ &\leq \lim_{n\to\infty}\int_{|x|\leq R_1}\int_{|v|\leq R_2} f_n dv\,dx+\frac{C}{R_2^2}\end{aligned}$$

where $R_1>R_0$ and $R_2>0$ are arbitrary, it follows that

$$\int_{|x|\leq R_1}\int f_0 dv\,dx=M$$

for every $R_1>R_0$. This proves the assertion on $\mathrm{supp} f_0$ and $\iint f_0=M$. Also by weak convergence

$$\iint |v|^2 f_0 dv\,dx\leq \liminf_{n\to\infty}\iint |v|^2 f_n dv\,dx<\infty. \tag{4.1}$$

By Lemma 1 $(\rho_n)=(\rho_{f_n})$ is bounded in $L^{1+1/n_1}(\mathbb{R}^3)$ where $n_1=\mu_1+3/2$. After extracting a further subsequence, we thus have that

$$\rho_n \rightharpoonup \rho_0 := \rho_{f_0} \text{ weakly in } L^{1+1/n_1}(\mathbb{R}^3).$$

Since $1+1/n_1>4/3$

$$\|\nabla U_n - \nabla U_0\|_{L^q(\mathbb{R}^3)} \le C,\ n\in\mathbb{N}$$

with some $q>12/5>2$ by Young's inequality. On the other hand, the compact embedding $W^{2,1+1/n_1}(B_R)\subset W^{1,1}(B_R)$ implies that

$$\|\nabla U_n - \nabla U_0\|_{L^1(B_R)} \to 0,\ n\to\infty$$

for any $R>0$. By the usual interpolation argument,

$$\|\nabla U_n - \nabla U_0\|_{L^2(B_R)} \to 0,\ n\to\infty,$$

but since by spherical symmetry,

$$\int |\nabla U_n - \nabla U_0|^2 dx \le \int_{|x|\le R} |\nabla U_n - \nabla U_0|^2 dx + 4\pi \frac{M^2}{R}$$

the convergence of the fields holds in $L^2(\mathbb{R}^3)$.

It remains to show that f_0 is actually a minimizer, in particular, $J(f_0)<\infty$ so that $f_0\in\mathcal{F}_M$. By Mazur's Lemma there exists a sequence $(g_n)\subset L^{p_1}(\mathbb{R}^6)$ such that $g_n\to f_0$ strongly in $L^{p_1}(\mathbb{R}^6)$ and g_n is a convex combination of $\{f_k \mid k\ge n\}$. In particular, $g_n\to f_0$ a. e. on $\mathbb{R}^6$. By (Q4) the functional

$$f\mapsto \iint Q(f,L)\,dv\,dx$$

is convex. Combining this with Fatou's Lemma implies that

$$\iint Q(f_0,L)\,dv\,dx \le \liminf_{n\to\infty} \iint Q(g_n,L)\,dv\,dx \le \limsup_{n\to\infty} \iint Q(f_n,L)\,dv\,dx.$$

Together with (4.1) this implies that

$$J(f_0)\le \lim_{n\to\infty} J(f_n) < \infty;$$

note that $\lim_{n\to\infty} J(f_n)$ exists. Therefore,

$$\mathcal{D}(f_0) = J(f_0) - \frac{1}{8\pi}\int |\nabla U_0|^2 \le \lim_{n\to\infty}\Big(J(f_n) - \frac{1}{8\pi}\int |\nabla U_n|^2\Big) = \mathcal{D}_M,$$

and the proof is complete. *QED*

Theorem 2 *Let Q satisfy the assumptions (Q1)–(Q5), and let $f_0 \in \mathcal{F}_M$ be a minimizer of $\mathcal{D}$. Then*

$$f_0(x,v) = \begin{cases} q(E_0 - E, L) & , E_0 - E > 0, \\ 0 & , E_0 - E \le 0 \end{cases}$$

where

$$E = \frac{1}{2}|v|^2 + U_0(x),$$

$$E_0 = \frac{1}{M} \int\int (\partial_f Q(f_0, L) + E)\, f_0 \, dv\, dx < 0,$$

U_0 is the potential induced by f_0, and q is as defined in (2.1). Moreover, f_0 is a steady state of the Vlasov-Poisson system.

Proof. Let f_0 be a minimizer. We shall use the standard method of Euler-Lagrange multipliers to prove the theorem. For any fixed $\epsilon > 0$ let $\eta : \mathbb{R}^6 \to \mathbb{R}$ be measurable, with compact support, spherically symmetric, and such that

$$|\eta| \le 1, \;\; \eta \ge 0 \text{ a. e. on } \mathbb{R}^6 \setminus \mathrm{supp f_0}, \;\; \epsilon \le \mathrm{f_0} \le \frac{1}{\epsilon} \text{ a. e. on } \mathrm{suppf_0} \cap \mathrm{supp}\eta.$$

Below we will occasionally argue pointwise on $\mathbb{R}^6$ so we choose a representative of f_0 satisfying the previous estimate pointwise on $\mathrm{supp} f_0 \cap \mathrm{supp}\eta$. For

$$0 \le h \le \frac{\epsilon}{2(1 + \|\eta\|_1)}$$

we define

$$g(h) = M \frac{h\eta + f_0}{\|h\eta + f_0\|_1}.$$

Clearly,

$$M - \frac{\epsilon}{2} \le \|h\eta + f_0\|_1 \le M + \frac{\epsilon}{2}.$$

On $\mathrm{supp} f_0$ we have

$$\frac{1}{4} f_0 \le g(h) \le 2 f_0 \tag{4.2}$$

provided $0<\epsilon<\epsilon_0$ with $\epsilon_0>0$ sufficiently small. Note that $g(0)=f_0$. We expand $\mathcal{D}(g(h))-\mathcal{D}(f_0)$ in powers of h. Obviously,

$$\begin{aligned}
&\mathcal{D}(g(h))-\mathcal{D}(f_0)\\
&\quad=\iint\Big(Q(g(h),L)-Q(f_0,L)\Big)\,dv\,dx+\frac{1}{2}\iint|v|^2(g(h)-f_0)\,dv\,dx\\
&\quad\quad-\frac{1}{8\pi}\int\Big(|\nabla U_{g(h)}|^2-|\nabla U_0|^2\Big)\,dx\\
&\quad=\iint\Big(Q(g(h),L)-Q(f_0,L)\Big)\,dv\,dx+\frac{1}{2}\iint|v|^2(g(h)-f_0)\,dv\,dx\\
&\quad\quad+\iint U_0(g(h)-f_0)\,dv\,dx-\frac{1}{8\pi}\int|\nabla U_{g(h)}-\nabla U_0|^2dx. \qquad (4.3)
\end{aligned}$$

Now observe that $g(h)\geq 0$ on $\mathbb{R}^6$. Thus $g(h)$ is differentiable with respect to h, and we write $g'(h)$ for this derivative. Note that both $g(h)$ and $g'(h)$ are actually functions of $(x,v)\in\mathbb{R}^6$, but we suppress this dependence. We obtain

$$\begin{aligned}
g'(h)&=\frac{M}{\|h\eta+f_0\|_1}\eta-M\frac{h\eta+f_0}{\|h\eta+f_0\|_1^2}\iint\eta\,dv\,dx,\\
g''(h)&=-2\frac{M}{\|h\eta+f_0\|_1^2}\left(\iint\eta\,dv\,dx\right)\eta+2M\frac{h\eta+f_0}{\|h\eta+f_0\|_1^3}\left(\iint\eta\,dv\,dx\right)^2.
\end{aligned}$$

Now

$$g'(0)=\eta-\frac{1}{M}\left(\iint\eta\,dv\,dx\right)f_0 \qquad (4.4)$$

and

$$|g''(h)|\leq C\,(|\eta|+f_0)$$

so that on $\mathbb{R}^6$,

$$|g(h)-f_0-hg'(0)|\leq C\,h^2(|\eta|+f_0);$$

in the following, constants denoted by C may depend on f_0, η, and ϵ but never on h. We can now estimate the last three terms in (4.3):

$$\iint|v|^2(g(h)-f_0)\,dv\,dx=h\iint|v|^2g'(0)\,dv\,dx+O(h^2), \qquad (4.5)$$

$$\iint U_0(g(h)-f_0)\,dv\,dx=h\iint U_0\,g'(0)\,dv\,dx+O(h^2), \qquad (4.6)$$

$$\begin{aligned}
\int|\nabla U_{g(h)}-\nabla U_0|^2\,dx&=\int|\nabla U_{g(h)-f_0}|^2dx\\
&\leq C\|\rho_{g(h)}-\rho_0\|_{6/5}^2\leq Ch^2. \qquad (4.7)
\end{aligned}$$

For the last estimate we used Young's inequality and the fact that

$$|\rho_{g(h)}(x)-\rho_0(x)| \leq Ch \int (|\eta|+f_0)(x,v)\,dv.$$

It remains to estimate the first term in (4.3). Consider first a point $(x,v) \in \mathrm{supp} f_0$ with $f_0(x,v) > 0$. Then

$$\begin{aligned} Q(g(h),L)-Q(f_0,L) &= \partial_f Q(f_0,L)(g(h)-f_0)+\frac{1}{2}\partial_f^2 Q(\tau,L)(g(h)-f_0)^2 \\ &= h\partial_f Q(f_0,L)g'(0)+h^2\frac{1}{2}\partial_f Q(f_0,L)g''(\theta) \\ &\quad +\frac{1}{2}\partial_f^2 Q(\tau,L)(g(h)-f_0)^2 \end{aligned}$$

where τ lies between $g(h)$ and f_0 and θ lies between 0 and h; both τ and θ depend on (x,v). Thus

$$\begin{aligned} &\Big|Q(g(h),L)-Q(f_0,L)-h\partial_f Q(f_0,L)g'(0)\Big| \\ &\qquad \leq C\partial_f Q(f_0,L)\,(|\eta|+f_0)\,h^2+C\partial_f^2 Q(\tau,L)(|\eta|^2+f_0^2)\,h^2. \end{aligned}$$

By (4.2), τ lies between $f_0/4$ and $2f_0$, so by iterating (Q5) a finite, h-independent number of times we find

$$\partial_f^2 Q(\tau,L) \leq C\partial_f^2 Q(f_0,L).$$

By (Q3) and (Q5),

$$\left(2^{1+1/\mu_3}-1\right)Q(f_0,L) \geq Q(2f_0,L)-Q(f_0,L) \geq \partial_f Q(f_0,L)\,f_0+C\partial_f^2 Q(f_0,L)\,f_0^2$$

and thus

$$\Big|Q(g(h),L)-Q(f_0,L)-h\partial_f Q(f_0,L)g'(0)\Big| \leq CQ(f_0,L)h^2+C|\eta|h^2;$$

here we used the continuity of $\partial_f Q$ and $\partial_f^2 Q$, the fact that $\epsilon \leq f_0 \leq 1/\epsilon$ on $\mathrm{supp}\eta \cap \mathrm{supp} f_0$, and the fact that L ranges in some compact interval if $(x,v) \in \mathrm{supp}\eta$. The above estimate holds for any point $(x,v) \in \mathrm{supp} f_0$ with $f_0(x,v) > 0$. Now consider a point (x,v) with $f_0(x,v)=0$. Then

$$g(h) = M\frac{h\eta}{\|h\eta+f_0\|_1} \leq C|\eta|h$$

so that by (Q4) and (Q2),

$$\Big|Q(g(h),L)-Q(f_0,L)-h\partial_f Q(f_0,L)g'(0)\Big|=Q(g(h),L)\le Q(Ch|\eta|,L)$$
$$\le C|\eta|^{1+1/\mu_2}h^{1+1/\mu_2}$$

for $h>0$ sufficiently small. Thus

$$\int\!\!\int\Big|Q(g(h),L)-Q(f_0,L)-h\partial_f Q(f_0,L)g'(0)\Big|\,dv\,dx\le Ch^{1+\delta} \tag{4.8}$$

for some $\delta>0$. Combining (4.5), (4.6), (4.7), and (4.8) with the fact that f_0 is a minimizer we find

$$0\le\mathcal{D}(g(h))-\mathcal{D}(f_0)=h\int\!\!\int\left(\partial_f Q(f_0,L)+\frac{1}{2}|v|^2+U_0\right)g'(0)\,dv\,dx$$
$$+O(h^{1+\delta})$$

for all $h>0$ sufficiently small. Recalling (4.4) and the definitions of E and E_0 this implies that

$$\int\!\!\int\Big(\partial_f Q(f_0,L)+E-E_0\Big)\eta\,dv\,dx\ge 0.$$

Recalling the class of admissable test functions η and the fact that $\epsilon>0$ is arbitrary, provided it is sufficiently small, we conclude that

$$E-E_0\ge 0\ \text{ a. e. on } \mathbb{R}^6\setminus\mathrm{supp f_0}$$

and

$$\partial_f Q(f_0,L)+E-E_0=0\ \text{ a. e. on } \mathrm{supp} f_0.$$

By definition of q—cf. (2.1)—this implies that

$$f_0(x,v)=q(E_0-E,L)\ \text{a. e. on } \mathbb{R}^6.$$

By construction,

$$\triangle U_0=\frac{1}{r^2}(r^2U_0')'=4\pi\rho_0$$

so that (f_0,ρ_0,U_0) is indeed a solution of the Vlasov-Poisson system. Since f_0 has compact support and $\lim_{r\to\infty}U_0(r)=0$ we conclude that $E_0<0$. *QED*

We conclude this section with a brief discussion of the uniqueness of the minimizer in $\mathcal{F}_M$. First observe that since each minimizer is spherically symmetric, has total mass M, support in B_{R_0}, and since $\lim_{r\to\infty} U_0(r) = 0$ we have

$$U_0(r) = -\frac{M}{r},\ r \geq R_0. \tag{4.9}$$

Since U_0 solves the ODE

$$\frac{1}{r^2}(r^2 U_0')' = 4\pi \int q\left(E_0 - \frac{1}{2}|v|^2 - U_0, L\right) dv$$

uniqueness would follow if E_0 were actually independent of the minimizer. If $Q(f) = f^{1+1/\mu}$ with $0 < \mu < 3/2$ the right hand side of the above ODE takes the form $c(E_0 - U_0)_+^{\mu+3/2}$ with some constant $c > 0$. Assume we have two solutions U_i with corresponding E_i, $i = 1,2$. Then in the terminology of [3], $\phi_i := E_i - U_i$ are E-solutions of the Emden-Fowler equation

$$\frac{1}{r^2}(r^2 \phi')' = -c(\phi)_+^{\mu+3/2}.$$

As is shown in [3], solutions of this ODE are turned into solutions of an autonomous, planar system by the change of variables

$$u(t) := -\frac{r\phi(r)^{\mu+3/2}}{\phi'(r)},\ v(t) := -\frac{r\phi'(r)}{\phi(r)},\ t = \ln r,$$

and the E-solutions are all mapped onto the same orbit, called C_3 in [3]. Reexpressed in terms of U this implies that

$$E_1 - U_1(r) = \gamma^{2/(\mu+1/2)}(E_2 - U_2(\gamma r)),\ r > 0,$$

for some $\gamma > 0$; note that a shift in t corresponds to a scaling in r. But then (4.9) implies $\gamma = 1$ and $U_1 = U_2$. We have not been able to find an analogous argument for Q's of a more general form.

5 Dynamical Stability

We now investigate the dynamical stability of f_0. First we note that for $f \in \mathcal{F}_M$,

$$\mathcal{D}(f) - \mathcal{D}(f_0) = d(f, f_0) - \frac{1}{8\pi}\|\nabla U_f - \nabla U_0\|_2^2. \tag{5.1}$$

where

$$d(f,f_0)=\int\!\!\int\Big[Q(f,L)-Q(f_0,L)+(E-E_0)(f-f_0)\Big]\,dv\,dx.$$

Theorem 3 *Let Q satisfy the assumptions (Q1)–(Q5) and assume that the minimizer f_0 is unique in $\mathcal{F}_M$. Then for all $\epsilon>0$ there is $\delta>0$ such that for any solution $f(t)$ of the Vlasov-Poisson system with $f(0)\in C^1_c(\mathbb{R}^6)\cap\mathcal{F}_{\mathrm{M}}$,*

$$d(f(0),f_0)+\frac{1}{8\pi}\|\nabla U_{f(0)}-\nabla U_0\|_2^2<\delta$$

implies

$$d(f(t),f_0)+\frac{1}{8\pi}\|\nabla U_{f(t)}-\nabla U_0\|_2^2<\epsilon,\ t\geq 0.$$

Proof. We first show that $d(f,f_0)\geq 0,\ f\in\mathcal{F}_M$. For $E-E_0\geq 0$ we have $f_0=0$, and thus

$$Q(f,L)-Q(f_0,L)+(E-E_0)(f-f_0)=Q(f,L)\geq 0.$$

For $E-E_0<0$,

$$Q(f,L)-Q(f_0,L)+(E-E_0)(f-f_0)=\frac{1}{2}\partial_f^2 Q(\bar{f},L)(f-f_0)^2\geq 0$$

provided $f>0$; here $\bar{f}$ is between f and f_0. If $f=0$, the left hand side is still nonnegative by continuity.

Now assume the assertion of the theorem were false. Then there exist $\epsilon_0>0$, $t_n>0$, and $f_n(0)$ such that $f_n(0)\in\mathcal{F}_M$, and

$$d(f_n(0),f_0)+\frac{1}{8\pi}\|\nabla U_{f_n(0)}-\nabla U_0\|_2^2=\frac{1}{n}$$

but

$$d(f_n(t_n),f_0)+\frac{1}{8\pi}\|\nabla U_{f_n(t_n)}-\nabla U_0\|_2^2\geq\epsilon_0>0.$$

From (5.1), we have $\lim_{n\to\infty}\mathcal{D}(f_n(0))=\mathcal{D}_M$. But $\mathcal{D}(f)$ is invariant under the Vlasov-Poisson flow, hence

$$\lim_{n\to\infty}\mathcal{D}(f_n(t_n))=\lim_{n\to\infty}\mathcal{D}(f_n(0))=\mathcal{D}_M.$$

Thus, $(f_n(t_n))\subset\mathcal{F}_M$ is a minimizing sequence of $\mathcal{D}$, and by Theorem 1 , we deduce that—up to a subsequence—$\|\nabla U_{f_n(t_n)}-\nabla U_0\|_2^2\to 0$. Again by (5.1), $d(f_n(t_n),f_0)\to 0$, a contradiction. *QED*

Corollary 1 *In addition to the assumption in Theorem 3 assume that*

$$C_1 := \inf \left\{ \partial_f^2 Q(f,L) \,|\, 0 < f \leq C_0,\ 0 \leq L \leq C_0 \right\} > 0$$

for some constant $C_0 > \|f_0\|_\infty + \max_{\mathrm{supp} f_0} L$*. If in the situation of Theorem 3* $f(0) \leq C_0$ *and* $L \leq C_0$ *on* $\mathrm{supp} f(0)$ *then*

$$\iint_{\mathbb{R}^6 \setminus \mathrm{supp f_0}} Q(f(t),L)\, dv\, dx + C_1 \iint_{\mathrm{supp} f_0} |f(t) - f_0|^2 dv\, dx$$
$$+ \frac{1}{8\pi} \|\nabla U_{f(t)} - \nabla U_0\|_2^2 < \epsilon,\ t \geq 0.$$

Proof. Since L is constant along characteristics and $\|f(t)\|_\infty = \|f(0)\|_\infty$,

$$f(t) \leq C_0,\ L \leq C_0 \text{ on } \mathrm{supp} f(t),\ t \geq 0.$$

Thus on $\mathrm{supp} f_0$,

$$Q(f(t),L) - Q(f_0,L) + (E - E_0)(f(t) - f_0) \geq C_1 |f(t) - f_0|^2,$$

cf. the argument in the proof of Theorem 3. *QED*

The assumptions of this corollary hold in particular for $Q(f,L) = f^{1+1/\mu}$ with $1 \leq \mu < 3/2$ and for linear combinations of such terms of the form (2.2).

Remark: Assume the minimizer f_0 of $\mathcal{D}$ is not unique in $\mathcal{F}_M$ and denote by $\mathcal{M}_M$ the set of all minimizers of $\mathcal{D}$ in $\mathcal{F}_M$. Then for each $\epsilon > 0$ there exists $\delta > 0$ such that for any solution $f(t)$ of the Vlasov-Poisson system with $f(0) \in \mathcal{F}_M \cap C_c^1(\mathbb{R}^6)$,

$$\inf_{f_0 \in \mathcal{M}_M} \left[d(f(0), f_0) + \frac{1}{8\pi} \|\nabla U_{f(0)} - \nabla U_0\|_2^2 \right] < \delta$$

implies

$$\inf_{f_0 \in \mathcal{M}_M} \left[d(f(t), f_0) + \frac{1}{8\pi} \|\nabla U_{f(t)} - \nabla U_0\|_2^2 \right] < \epsilon,\ t \geq 0.$$

The proof works along the same lines as for Theorem 3.

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